Are Control Density Functions Practical?

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Introduction

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- Safety-critical control seeks to ensure autonomous systems **avoid unsafe regions**
- Control Barrier Functions (CBFs): construct geometric safe sets; enforce invariance through **gradient-based** conditions
- Control Density Functions (CDFs): similar to CBFs, but reasons about **divergences**

Questions

1. Are there systems where CDFs simplify **safety verification**?
2. Are CDFs a practical alternative to CBFs for **controller synthesis**?

Safety Conditions

Both CDFs and CBFs use $\rho : \mathcal{X} \to \mathbb{R}$ satisfying:

1. $\rho(x) \geq 0$ on the initial set $\mathcal{X}_0$
2. $\rho(x) < 0$ on the unsafe set $\mathcal{X}_u$
3. $\underbrace{D_{\text{DF}}(x) \geq 0}_{\text{for CDFs}}$ **or** $\underbrace{D_{\text{BF}}(x) \geq 0}_{\text{for CBFs}}$ on $\mathcal{X}$

where

$$
\begin{aligned}
D_{\text{BF}}(x) &= \nabla\rho(x)^\top f(x) + \alpha\rho(x) \\
D_{\text{DF}}(x) &= [\nabla \cdot (\rho f)](x) + \alpha\rho(x) \\
&= [\nabla \cdot f](x)\,\rho(x) + D_{\text{BF}}(x)
\end{aligned}
$$

Key observation

CDFs and CBFs **differ only by the divergence** term $(\nabla \cdot f)\,\rho$.

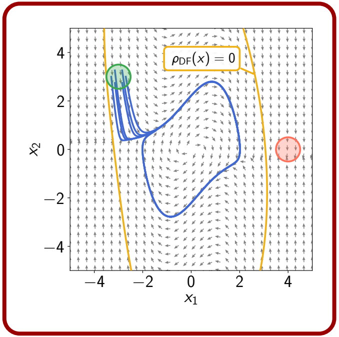
The Certificate Problem

Divergence term has **ambiguous sign** (system-dependent):

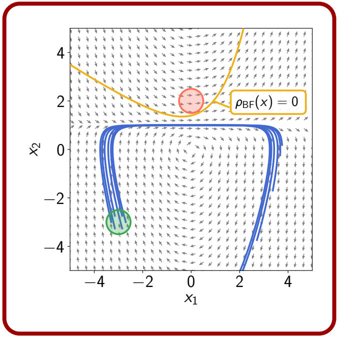
- Helps when $(\nabla \cdot f)\,\rho > 0 \Rightarrow$ CDF easier than CBF
- Hinders when $(\nabla \cdot f)\,\rho < 0 \Rightarrow$ CBF easier than CDF

Demonstrated via sum-of-squares programming on two systems:

CDF ✓ CBF ✗



CDF ✗ CBF ✓



Green: initial set $\mathcal{X}_0$

Red: unsafe set $\mathcal{X}_u$

Blue: trajectories from $\mathcal{X}_0$

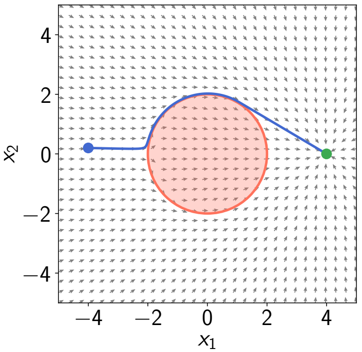
Yellow: zero level set of ρ

Synthesis: CDFs = CBFs?

- **Goal:** find $u^\star$ that minimally modifies u_{ref}
- For CBFs, it easy to formulate a QP; can we do something similar for CDFs?
- Divergence introduces $\frac{\partial u}{\partial x} \Rightarrow$ **QP hard to form**
- Restrict to polynomial controllers $u(x) = Q(x)b$:

$$b^\star(x) = \arg\min_b \quad \|Q(x)b - u_{\text{ref}}(x)\|^2 + \frac{\lambda}{2}\|b\|^2$$
$$\text{s.t.} \quad d(x)^\top b + c(x) \geq 0.$$

- $b^\star(x)$ is state-dependent $\Rightarrow$ CDF condition is **not satisfied in general**
- On the boundary $\partial\mathcal{X}_u,\ \rho(x) = 0,$ so $D_{\text{DF}}(x) = D_{\text{BF}}(x)$
- Safety follows from **Nagumo-style CBF arguments**



Polynomial CDF controller for a single-integrator: x_0 (blue), x_T (green), $\mathcal{X}_u$ (red).

Conclusion

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- CDFs and CBFs differ only by $(\nabla \cdot f)\rho$; sign ambiguity complicates safety verification
- CDF synthesis reduces to the same conditions as CBFs in practice
- Density functions may help with verification, but are unlikely to simplify controller synthesis

pauker.org/posts/l4dc2026

Are Control Density Functions Practical?

For safety verification: possibly. For controller synthesis: probably not.